International Journal of Theoretical Physics, Vol. 35, No. 12, 1996

Proton Dipole Form Factor Quark Model

George L. Strobel[1]

Received January 11, 1996

The dipole-shaped electromagnetic form factors of the proton may imply an exponential radial dependence of the wave function describing the charged constituents of the proton. The hypercentral potential required by the three-body Dirac equation to produce such an exponential radial wave function for three bound quarks is found to have a linear confining potential plus an attractive Coulombic central diagonal part. The configuration assumed for the quark constituents is the $(1/2^+)^3$ positive parity configuration, coupled to the spin of the proton. Assuming equal-mass Dirac quarks with no anomalous magnetic moments, we find the largest magnetic moment for this wave function to be 2.763 nuclear magnetons, close to, but less than the experimental value of 2.793. The hypercentral potential is mostly the sum of three quark–quark potentials, but a small three-body potential is required.

1. INTRODUCTION

The proton stability, the nondiscovery of free quarks, and considerations of simplicity have led to harmonic oscillator interactions and Gaussian-type wave functions in numerous calculations (Feynman *et al.*, 1971; Capstick and Isgur, 1986; Fleck *et al.*, 1988; Gutsche *et al.*, 1994; Grach and Narodetskii, 1994; Filippov *et al.*, 1994). The form factor, which is the momentum Fourier transform, of such a Gaussian wave function is also a Gaussian versus q^2. However, the electromagnetic form factors can be well fit (West, 1975) by a dipole form factor, suggesting an exponential shape for the charge distribution in the proton, in sharp contrast to a Gaussian form factor. The dipole form factor is

$$\text{Dipole} = 1./(1. + q^2/0.71)^2 \tag{1}$$

where q^2 is the squared momentum transfer to the proton in $(\text{GeV/c})^2$. This form factor very nearly describes all the momentum dependences of both

[1] Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602; e-mail: gstrobel@hal.physast.uga.edu.

0020-7748/96/1200-2443$09.50/0

GM and GE, the current and charge form factors (Halzen and Martin, 1984) of the proton.

In this work, the proton is assumed to be composed of three identical, nearly massless quarks. The quarks are considered as elementary constituents without any anomalous magnetic moment nor any finite size. The three-body Dirac equation, including the lower components to the quark wave functions, provides a rest-frame wave function determined by relativistic dynamics.

Barut and Komy (1985) have shown how to get a Poincaré invariant n-body equation with a single time where the relative coordinates depend only on three vectors in the rest frame of the cluster. They introduce derivatives with respect to the composite fields of two (or n) fermions and take derivatives of the action with respect to these fields using retarded Green's functions. This results in a covariant two (or n)-body one-time equation with relativistic potentials. The time is the time of the center of mass. In a three-body system there is no dependence on either relative time. When the two (or n)-body composite particle wave function normalization is defined on a surface perpendicular to a unit four-vector (Barut and Strobel, 1986), which is (1, 0, 0, 0) in the overall center-of-momentum frame, then there is no relative time in the composite particle Hamiltonian in this frame. This unit normal vector, perpendicular to the surface over which the normalization is determined, can be identified with the frame-dependent velocity vector of the system (Moreno and Zentella, 1989), which is also (1, 0, 0, 0) in the center-of-momentum frame. This allows the rest-frame wave function to be expressed covariantly in terms of the three vector relative coordinates.

The noninteracting form of the three-body Dirac equation for equal quark masses M is

$$[\alpha_1 \bullet (2P_1 - P_2 - P_3) + \alpha_2 \bullet (-P_1 + 2P_2 - P_3) + \alpha_3 \bullet (-P_1 - P_2 + 2P_3) + 3M(\beta_1 + \beta_2 + \beta_3)]\Psi/3 = E\Psi \quad (2)$$

α and β are the Dirac matrices, and P is the vector momentum operator. The subscripts denote particle label. This is the so-called Hamiltonian form. The hyperspherical method has been applied to this three-body Dirac equation (Strobel, 1986), where hyperangular averages of a diagonal central potential and the relativistic kinetic energy operator were evaluated. The basic idea is to use the chain rule of calculus to change the partial derivatives of the kinetic energy operator with respect to r_1, etc., into partial derivatives with respect to the hyperradius.

The hyperradius is defined as

$$\rho^2 = (r_{12}^2 + r_{23}^2 + r_{31}^2)/3 = r_1^2 + r_2^2 + r_3^2 - 3R^2 \quad (3)$$

where r_1, r_2, and r_3 are the locations of the three particles, respectively, and R is the location of the center of mass. r_{12} is the separation of particles 1 and 2, etc. The hyperspherical formulation expands the three-body bound-state wave function into a set of terms, each of which has a hyperradial and a hyperangular factor. For a hypercentral potential there is only a single term in the expansion. The details of the hyperspherical approach can be found in book Morse and Feshbach (1953) or Baz and Zhukov (1970).

A hypercentral potential will be added to this free three-body Dirac equation to bring about quark confinement and an exponentially decreasing hyperradial wave function. That such a hypercentral potential can be analytically found for the three-body Dirac equation is the major result of this paper. One could just add an arbitrary three-body hypercentral potential. Instead, the hypercentral potential found below will be considered as the sum of three pairwise quark–quark potentials. These quark–quark potentials are taken as diagonal central potentials such that the quark–antiquark two-body system is confined (Stanley and Robsen, 1980) and avoids the Klein paradox (Semay *et al.*, 1993). The hypercentral potential found below cannot be described as solely the sum of three pairwise two-body potentials. The pairwise potentials by themselves nearly reproduce the hyperradial potential, but a small residual three-body potential is also required.

2. THEORY

The three-fermion wave function Ψ is written as

$$\Psi = \Sigma\, R(\rho) U(\Omega) \tag{4}$$

where the sum in general is over various configurations. U is a product of the spin, flavor, and color parts of the wave function for each of the particles, and includes the angular momentum coupling. Ω denotes the hyperangles and the other spin, flavor, and color coordinates of the system.

The unknown hyperradial dependence to be determined is contained in the $R(\rho)$ factor, which is represented here as an eight-component vector. The angular momentum coupling is $[j_1, j_2]J_{12}, j_3 J M_z\rangle$. Here j_1, j_2, and j_3 are the total angular momentum of each of the three particles; J_{12} is the intermediate coupling of the first pair. The total angular momentum of the third particle is coupled to J_{12} to produce J, the total angular momentum of the three-body system, and its z component M_z. For the nucleon, J is one-half. Doing the hyperangular integration results in the three-body Dirac equation becoming a set of coupled differential equations involving derivatives with respect to the hyperradius. The summation over configurations is restricted here as follows. Each term in the summation is a given spin parity configuration. The only spin parity configuration considered is the $(1/2^+)^3$. For this configuration an eight by eight matrix is obtained for the Hamiltonian operating on the composite three-body wave function involving the large and small components for each particle. There are eight components of the composite three-body Dirac equation wave function for a given configuration, denoted by an x subscript. Each component has a K value which is the sum of the orbital angular momentum for each of the three single-particle orbitals of that component. K depends on x, and is tabulated below. We define $\Lambda(x, x') = K(x) + K(x')$. This angular momentum coefficient varies from element to element in the Hamiltonian matrix. The normalization of the wave function is now considered. The norm for a given single configuration is

$$1 = \sum_x \int \rho^5 \, d\rho (R_x)^2 / \Gamma[\Lambda(x, x) + 6)/2] \tag{5}$$

Here the sum is over the eight components of the single configuration of the composite three-body wave function.

After integrating over the hyperangles, the three-body Dirac equation is found to be a set of one-dimensional differential equations in the hyperradius. The kinetic energy operator depends on whether it operates on a large or on a small component single-particle wave function (Strobel and Hughes, 1987). The kinetic energy operator is $-\rho^{K(x')}D(-K(x))$. Here the quantity D is defined as

$$D(n) = \sqrt{(2/3)}\,(d/d\rho + n/\rho) \tag{6}$$

The somewhat ugly factor of $\sqrt{(2/3)}$ can be eliminated through a change of variables by defining

$$3\rho^2 = 2r^2 \tag{7}$$

In the equations that follow, r refers to the square root of $3\rho^2/2$.

3. AN EXPONENTIAL SOLUTION FOR THE $(1/2^+)^3$ CONFIGURATION

For the $(1/2^+)^3$ configuration the K values are as follows:

x	$L1$	$L2$	$L3$	K
1	0	0	0	0
2	1	0	0	1
3	0	1	0	1
4	1	1	0	2
5	0	0	1	1
6	1	0	1	2
7	0	1	1	2
8	1	1	1	3

An exponential solution for the hyperangular integrated form of $[H - E]\Psi = 0$ for the $(1/2^+)^3$ configuration is found when a specific hypercentral potential is utilized for the case of all three particles identical. For three identical particles, and with each particle with the same set of quantum numbers, one expects the components $R2$, $R3$, and $R5$ to be equal, and also for the components $R4$, $R6$, and $R7$ to be equal. Then the wave function has only four unknown components, $R1$, $R2$, $R4$, and $R8$. The hypercentral interaction can vary from component to component, but for three identical particles, the interactions in components 2, 3, and 5 are equal. Also, the interaction in components 4, 6, and 7 are equal. Including a hypercentral interaction along the diagonal results in a 4 by 4 Hamiltonian matrix that operates on the four unknown components as

$$\begin{bmatrix} (3M - E + V_{hs1}) & -D(5) & 0 & 0 \\ D(0) & (M - E + V_{hs2}) & -D(6)/2 & 0 \\ 0 & 2D(-1) & (-M - E + V_{hs4}) & -D(7)/5 \\ 0 & 0 & 3D(-2) & (-3M - E + V_{hs8}) \end{bmatrix} \begin{bmatrix} R1 \\ R2 \\ R4 \\ R8 \end{bmatrix} = 0 \quad (8)$$

This matrix operates on the hyperradial components $R1$, $R2$, $R4$, and $R8$. Solutions of this equation are sought and found of the form

$$\begin{aligned} R1 &= A \exp(-Lr) \\ R2 &= Br \exp(-Lr) \\ R4 &= Cr^2 \exp(-Lr) \\ R8 &= Hr^3 \exp(-Lr) \end{aligned} \quad (9)$$

The powers of r for each of the components handle the dominant angular momentum barrier at the origin. The exponential factor for each of the components comes from the dipole fit to the proton form factors suggesting an exponential charge distribution. L is an inverse length parameter that will be determined from physical considerations. The component coefficients found for the composite wave function are

$$\begin{aligned} A &= 1 \\ B &= -(E - 3M)/6 \\ C &= (E - M)(E - 3M)/24 \\ H &= -(E + M)(E - M)(E - 3M)/48 \end{aligned} \quad (10)$$

An exponential solution of this form is possible only when the hypercentral potential parameters are chosen such that

$$V_{hsx} = n_x r + k_x + m_x/r \quad (11)$$

where

$$\begin{aligned} n_1 &= L(E - 3M)/6, & k_1 &= 0 & m_1 &= 0 \\ n_2 &= L(E - 3M)/8, & k_1 &= 0 & m_2 &= -6L/(E - 3M) \\ n_4 &= L(E + 3M)/10, & k_4 &= 0 & m_4 &= -8L/(E - M) \\ n_8 &= 0 & k_8 &= (E + 3M), & m_8 &= -6L/(E + M) \end{aligned} \quad (12)$$

This hypercentral potential is determined when the exponential solution is substituted into the four by four Hamiltonian matrix, the differentiation is carried out, and coefficients of like powers of r are equated. Setting the coefficient A to unity, we obtain the above solution and potentials. Normalizing this configuration to unity multiplies all the component coefficients by a constant compared to the values quoted above.

L is a parameter of the exponential wave function, and the hypercentral potential parameters depend on it. The larger L is, the more pointlike the system becomes. The system is confined, as the parameters n_x are all nonnegative. It is interesting that the $1/\rho$ coefficients are all nonpositive. This is reminiscent of an attractive one-gluon exchange potential contribution at short distances and a long-range linear confining potential at large distances.

The contribution of each component of the wave function to the normalization varies with a relativistic parameter T, defined as

$$T = (E - 3M)/(E + 3M) \quad (13)$$

This parameter is one in the extreme relativistic case where the quark mass M is negligible compared to the proton rest energy E. The relativistic parameter is zero in the nonrelativistic limit where the mass is one-third of the energy. The magnetic moment for this wave function is a monotonically increasing function of T for all values of L. The calculated magnetic moment, for T equals unity, as a function of L can be seen in Fig. 1. The maximum value is 2.763 nuclear magnetons for an L value of 1.38 (1/fermi). For larger values of L the entire system shrinks, causing the expectation value of r to decrease, and also the magnetic moment to decrease. For smaller values of L, although the system expands, as does the expectation value of r, the system becomes dominated by the small–small–small component of the composite wave function, so much so that the magnetic moment matrix element decreases. This element has large components of the composite wave function entering linearly, and these components become negligible in the small-L limit for zero-mass quarks. The large–large–large component of the composite wave function dominates the normalization for large L, as can be seen in Fig. 2.

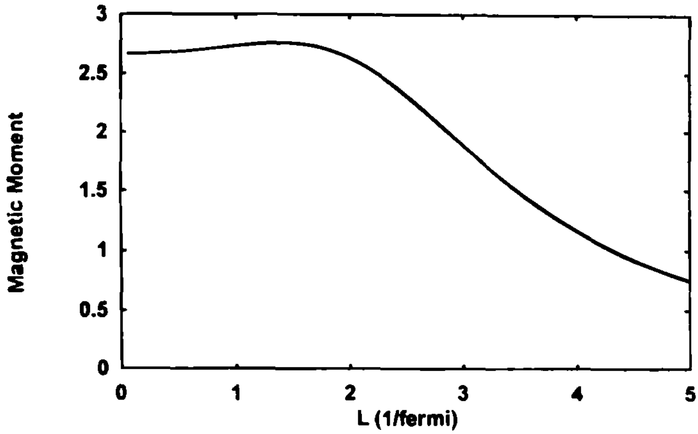

Fig. 1. Calculated proton magnetic moment for zero-mass Dirac quarks versus inverse size parameter L, for an exponential composite three-body wave function.

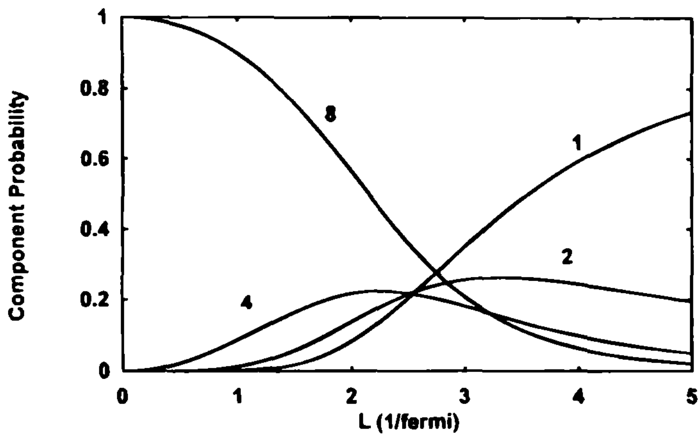

Fig. 2. The $R1$, $R2$, $R4$, and $R8$ component contributions to the $(1/2^+)^3$ configuration normalization versus the inverse size parameter L. The curves are for E equal to the proton rest energy. The $R2$ normalization includes the equal contributions from each of the $R2$, $R3$, and $R5$ components. The $R4$ normalization includes the equal contributions from the $R4$, $R6$, and $R7$ components. Zero-mass quarks are assumed. The curves are labeled by the component x value.

The components of the composite three-body wave function can be seen in Fig. 3 for the L value that maximizes the magnetic moment. The small cubed component dominates the normalization.

4. PAIRWISE POTENTIALS

The hypercentral potential found to produce an exponential hyperradial wave function cannot be described as solely from the sum of three pairwise

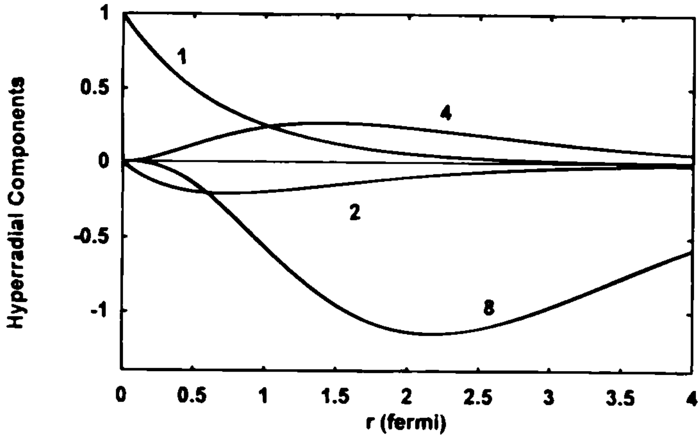

Fig. 3. Components of the composite three-body exponential wave function versus r. The hyperradius is related as $2\rho^2 = 3r^2$. The curves are for an inverse size parameter L of 1.38 1/Fermi. The curves are labeled by the component x value.

diagonal central potentials between identical quarks. A three-body potential is also needed.

The three-body potential contribution to the hypercentral potential V_{hsx} can be restricted to the $x = 1$ large cubed component of the composite three-body wave function or to the $x = 8$ small cubed component. The former is done here, as the confining pairwise potentials then obtained properly avoid the Klein paradox (Semay *et al.*, 1993). The corresponding quark–antiquark potentials do confine the two-body system. Restricting the three-body potential to the small cubed component results in two-body potentials that suffer from the Klein paradox.

The two-body central potential considered here is a central, diagonal interaction:

$$V_{12} = S_1\beta_1 + S_2\beta_2 + S\beta_1\beta_2 + V \tag{14}$$

with S_1, S_2, S, and V being in general (Semay *et al.*, 1993) unknown functions of r_{12}. This is a two-body generalization of the central diagonal scalar plus vector one-body potential used to solve the one-body Dirac equation. This is still not the most general possible two-body covariant potential, as no nondiagonal Dirac matrices are included. The radial part of a vector potential, which conserves parity, angular momentum, and time reversal invariance, drops out of a Schrödinger-like equation in a nonrelativistic reduction (Clark *et al.*, 1982; Miller, 1975). This and simplicity are the only reasons for considering diagonal Dirac matrices only. In the space of the large and small components for two bodies (FF, GF, FG, GG), where F now refers to the large component, and G refers to the small component, and ordering in a term refers to particle label, this two-body potential has only diagonal nonvanishing matrix elements. They are

$$\begin{Bmatrix} me_1 = +S_1 + S_2 + S + V & \text{FF} \\ me_2 = -S_1 + S_2 - S + V & \text{GF} \\ me_3 = +S_1 - S_2 - S + V & \text{FG} \\ me_4 = -S_1 - S_2 + S + V & \text{GG} \end{Bmatrix} \tag{15}$$

The quark–quark potential radial dependence, when integrated over hyperangles, that can best reproduce the hypercentral potential has the form

$$W_{12} = A_1 r_{12} + A_0 + A_m/r_{12} \tag{16}$$

This form is similar to other typical quark potentials (Nag *et al.*, 1987; Strobel and Pfenninger, 1987; Burov and Shitikova, 1992). With properly chosen coefficients A_1, A_0, and A_m, the sum of three of these pairwise potentials can be made to nearly reproduce the hypercentral potential. The match is complete for components $x = 2$–8. The deviation for component $x = 1$ can be attributed to a three-body potential. The quark mass is set to zero now, as this value results in the most realistic (largest) values for the model magnetic moment. The constant coefficient k_x for the hyperradial potential implies that the constant coefficient for the pairwise potential is given by

$$A_0 = E\begin{Bmatrix} 1/3 & \text{FF} \\ -1/6 & \text{GF} \\ -1/6 & \text{FG} \\ 1/3 & \text{GG} \end{Bmatrix} \tag{17}$$

A linear confining term to the pairwise potential is inferred from the part of the hyperradial potential V_{hsx} that is proportional to r. For the $A_1 r_{12}$ term to the W_{12} pairwise potential, we have

$$A_1 = cLE\begin{Bmatrix} 0.01250 & \text{FF} \\ 0.04557 & \text{GF} \\ 0.04557 & \text{FG} \\ 0.0 & \text{GG} \end{Bmatrix} \tag{18}$$

Here $c = 9\pi/16\sqrt{2}$. This factor comes from the hyperangular integration of r_{12} with the result proportional to r.

Likewise the A_m/r_{12} term of the pairwise potential is found to be

$$A_m = 2cL/E\begin{Bmatrix} -0.1597 & \text{FF} \\ -0.9418 & \text{GF} \\ -0.9418 & \text{FG} \\ -0.5833 & \text{GG} \end{Bmatrix} \tag{19}$$

The $2c$ coefficient appearing here comes from the hyperangular integration

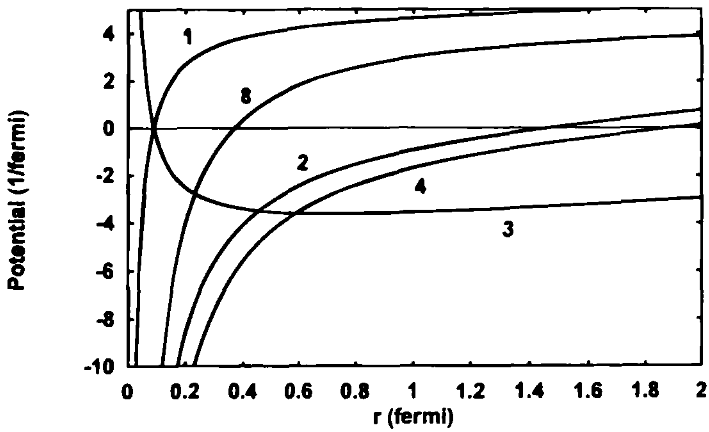

Fig. 4. The required hyperradial potentials acting on each component of the composite three-body wave function. The labels 1, 2, 4, and 8 denote the component x value on which the two-body potential acts. The three-body potential acting only on the large cubed component is labeled 3.

of $1/r_{12}$, with a result proportional to $1/r$. The hyperangular average of the sum of these three pairwise potentials W_{12}, W_{13}, and W_{23} is given as follows:

x	confinement	constant	Coulombic
1	$0.045LEr$	E	$-1.4375L/Er$
2	$LEr/8$	0	$-6L/Er$
4	$LEr/10$	0	$-8L/Er$
8	0	E	$-6L/Er$

(20)

The remainder of the required hyperradial potential is ascribed to a three-body potential of the form

$$V_{3b} = (0.12166LEr - E + 1.4375L/Er)(1 + \beta_1)(1 + \beta_2)(1 + \beta_3)/8 \tag{21}$$

Figure 4 shows these potentials as a function of r. This three-body potential, by itself, assists confinement. It acts only in the large cubed component of the wave function. It is a small part of the potential that confines the system. The expectation value of the three-body potential, for the case of $L = 1.38$ (1/fermi) is -0.0069 GeV. This is so small because the component where the three-body potential acts is such a minor component of the system normalization. The quark–quark potentials inferred from the hypercentral potential avoid the Klein paradox (Semay *et al.*, 1993). The expectation value of the pairwise potentials is 0.7546 GeV. The expectation value of the kinetic energy of the system is 0.1905 GeV. A three-body potential may be expected (Stanley and Robsen, 1980) for a three-quark system if the quark–quark potential is properly described by the nonlinear QCD theories. The nonlinear effects may be acting to prevent three pairwise potentials alone from describing the potentials acting in a three-quark system.

5. CONCLUSIONS

A hypercentral potential of the form

$$V_{hsx} = n_x r + k_x + m_x/r \tag{22}$$

has been found that will produce an exponentially damped solution for the three-body Dirac equation. For the $(1/2^+)^3$ configuration of quarks coupled to a spin of 1/2, the maximum magnetic moment is found to be 2.763 nuclear magnetons. This is for three massless Dirac quarks. The hypercentral potential can be mostly reproduced by the sum of three pairwise interactions integrated over hyperangles. The difference is attributed to a three-body potential. This is expected if the quark–quark potential is a nonlinear result of interacting gluon exchange. The pairwise potentials deduced are similar to those used in quark–antiquark descriptions of mesons. The required potentials have a Coulomb attraction, a constant term, and a linear confining term. An exponential wave function shape might be expected from the dipole fits to the electromagnetic form factors.

REFERENCES

Barut, A. O., and Komy, S. (1985). *Fortschritte der Physik*, **33**, 309.

Barut, A. O., and Strobel, G. L. (1986). *Few Body Systems*, **1**, 167.

Baz, A. I., and Zhukov, M. V. (1970). *Soviet Journal of Nuclear Physics*, **11**, 435.

Burov, V. V., and Shitikova, K. V. (1992). *Soviet Journal of Nuclear Physics*, **55**, 528.

Capstick, S., and Isgur, N. (1986). *Physical Review D*, **34**, 2809.

Clark, B. C., Hama, S., and Mercer, R. L. (1983). In *The Interaction between Medium Energy Nucleons in Nuclei—1982*, H. O. Meyer, ed., AIP, New York, p. 284.

Feynman, R. P., Kislinger, M., and Ravndal, F. (1971). *Physical Review D*, **3**, 2706.

Filippov, G. F., Nesterov, A. V., Rybkin, I. Yu., and Korennov, S. V. (1994). *Physics of Particles and Nuclei*, **25**(6), 569(1347).

Fleck, S., Silverstre-Brac, B., and Richard, J. M. (1988). *Physical Review D*, **38**, 1519.

Grach, I. L., and Narodetskii, I. M. (1994). *Few Body Systems*, **16**, 151.

Gutsche, T., Viollier, R. D., and Faessler, A. (1994). *Physics Letters B*, **331**, 8.

Halzen, F., and Martin, A. D. (1984). *Quarks and Leptons*, Wiley, New York.

Miller, L. D. (1975). *Annals of Physics*, **91**, 40.

Moreno, M., and Zentella, A. (1989). *Journal of Physics*, **22A**, L821.

Morse, P. M., and Feshbach, H. (1953). *Methods of Theoretical Physics*, McGraw-Hill, New York.

Nag, R., Sanyal, S., and Mukherjee, S. M. (1987). *Physical Review D*, **36**, 2788.

Semay, C., Ceuleneer, R., and Silvestre-Brac, B. (1993). *Journal of Mathematical Physics*, **34**, 2215.
Stanley, D. P., and Robsen, D. (1980). *Physical Review Letters*, **45**, 235.
Strobel, G. L. (1986). *Hadronic Journal*, **9**, 181.
Strobel, G. L., and Hughes, C. A. (1987). *Few Body Systems*, **2**, 155.
Strobel, G. L., and Pfenninger, T. (1987). *Physics Letters B*, **195**, 7.
West, G. B. (1975). *Physics Reports*, **18**, 263.